**Oscillating Cart: A Study of Damped Harmonic Motion**

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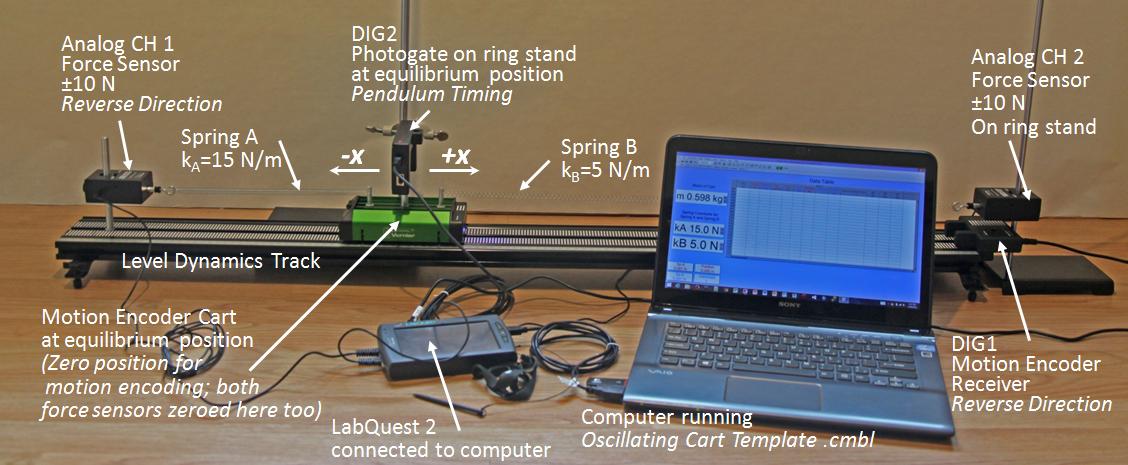
***Introduction***

After pumping a swing so that it is going back-and-forth very high, a boy stops pumping; the swing’s amplitude slowly decreases until it comes to rest. The boy’s grandfather, rocking back-and-forth in a rocking chair on the porch, falls asleep; the rocking chair gradually slows down until the rocking is unnoticeable. Meanwhile, the grandfather’s granddaughter, practicing the piano in the living room, taps the middle C key while pressing the sustain pedal with her foot; she notices that the sound eventually fades away until it is imperceptible.

These are but a few examples of damped harmonic motion in everyday life. Because such motion is so common, a quantitative study of it is typically a part of any physics course. Students can study the decrease in amplitude of a pendulum after it is released. The amplitude of up-and-down vibration of a paper plate attached to a weight hanging from a spring can be studied. A cart with a spring on each end, oscillating back-and-forth on a track, offers a particularly exciting way for the physics student to study damped harmonic motion. The Vernier Motion Encoder System, Dual-Range Force Sensors, Photogate, and Springs Set used in conjunction with a LabQuest 2 and computer running Logger *Pro* for data collection make the oscillating cart experiment easy to perform with precision motion measurement.

***The Experiment Set-Up***

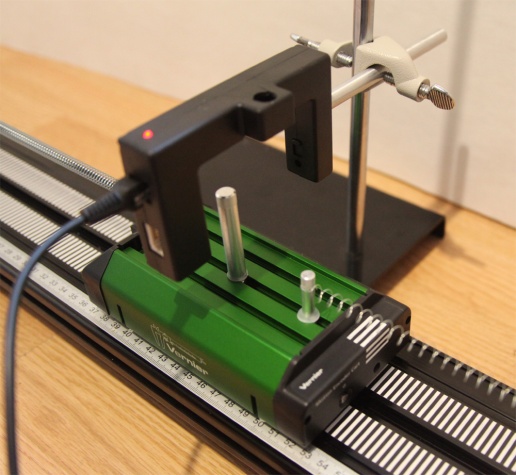
Figure 1 shows the setup for the Vernier oscillating cart experiment for studying damped harmonic motion.



*Figure 1*

The Vernier Spring Set includes three 5 N/m and three 15 N/m springs, so a variety of combinations for the two springs on either side of the cart is possible. Figure 1 shows that Spring A on the left is a 15 N/m spring while Spring B on the right is a 5 N/m spring. The springs are attached so that they are visually parallel to the track and, hence, as level as possible. The Dual-Range Force Sensor on the right is attached to a ring stand since the rod used to attach it to the Dynamics Track interferes with the IR signal sent from the Motion Encoder Cart to the Motion Encoder Receiver. You will probably want to clamp the ring stand to the edge of a lab table so that it will not move under the tension of the spring. The Motion Encoder Cart is shown resting at its equilibrium position, in which the forces on the cart for each spring are equal in magnitude but oppositely directed. The Dual-Range Force Sensors are zeroed when the cart is at equilibrium, with the force sensor at the left end of the track set for *Reverse Direction* since Spring A is always pulling the cart to the left. With the cart at its equilibrium position, the Motion Encoder Receiver is zeroed and set for *Reverse Direction*. As a result, the equilibrium position corresponds to *x*=0, with positive *x* to the right and negative *x* values to the left. As the cart is gradually moved to the right of the equilibrium position, the Spring A force becomes more and more negative since Spring A is being stretched more. Meanwhile, the Spring B force becomes more and more negative since its force on the cart is being *reduced*. Similar logic can be applied when the cart is gradually moved to the left of the equilibrium position. A Photogate (internal gate mode) at the equilibrium position, set for *pendulum timing*, is used to determine the period of the cart as it oscillates back-and-forth on the leveled Dynamics Track. The various sensors are connected to the LabQuest 2 input channels as indicated in Figure 1. LabQuest 2 is in turn connected to a computer via the computer connection (USB) cable. The computer is running the file *Oscillating Cart Template.cmbl* (included as part of the supporting documentation for this experiment). This Logger *Pro* file will automatically collect data and provide graphs for position, velocity, acceleration, spring forces, period, kinetic energy, potential energy, and mechanical energy—all ready for a detailed analysis by the student.

Figure 2 shows a close-up of the cart at its equilibrium position. A tall aluminum peg is mounted at the center of the cart to interrupt the IR beam as the cart moves back-and-forth. Short aluminum pegs are used on each side of the cart to attach the springs to the cart. Students should be cautioned to use care when connecting the springs to the cart. It is also advisable for them to wear goggles since stretched springs may snap.



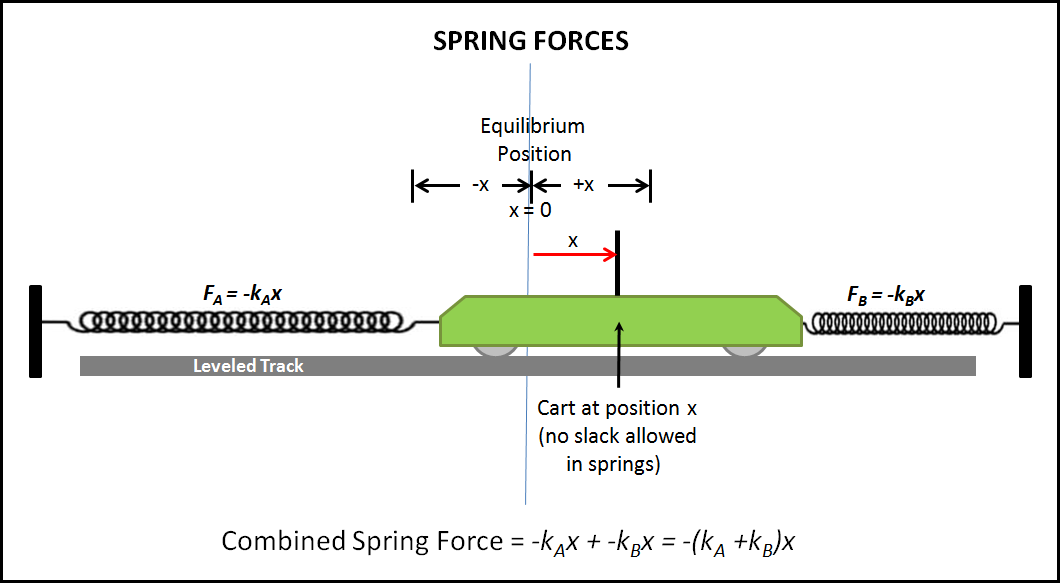
*Figure 2*

***Oscillating Cart System Forces***

Figure 3 shows the cart system and the spring forces acting on the cart. In effect, the two separate springs with spring constants *kA* and *kB* act like a single spring whose spring constant is *kA + kB*. With the cart oscillating on the leveled track, the normal force and the force of gravity balance one another. The remaining forces, in addition to the force from the springs, are the damping forces that cause the cart to slow down as time progresses during oscillation.

These damping forces are opposite to the velocity vector at all times during the cart’s oscillation. Students should be encouraged to suggest sources for these damping forces. Some possible damping forces include rolling friction, bearing friction of the wheels on their axles, losses due to heating of the springs while being stretched and unstretched, viscous damping from air resistance, and losses due to possible deformation of the springs.

Note that the 15 N/m springs in the Vernier Springs Set may deform when stretched more than about 0.25 meter, whereas the 5 N/m springs can be stretched on the order of 0.7 meter without deformation. Students should be cautioned not to damage the springs by stretching them beyond these limits.



*Figure 3*

Another source of damping that becomes particularly noticeable if rubber bands are used instead of the springs is hysteresis damping. (See Elastic Hysteresis of a Rubber Band at the following URL: <http://www.vernier.com/innovate/elastic-hysteresis-of-a-rubber-band/>.) You can have students perform the oscillating cart experiment with rubber bands as well as springs. Suitable latex free, stretchable bands that are 7 inches long can be purchased from most of the well-known office supply stores. These bands have spring constants on the order of 25 N/m. The oscillations will die down much more rapidly than with the springs. Students may find that the equilibrium position has changed from its initial position following collection of data and should be asked to provide possible explanations for this.

***A Model for Damping***

In the previous section a variety of reasons were presented for the loss of energy during the damping process. Although the forces involved in damping are quite complex, there is a *linear drag model* that accurately describes a great majority of damped harmonic motions. This model assumes that the damping force is proportional to, but in a direction opposite to that of the velocity, that is *F = -bv*, where *b* is a positive constant known as the *damping factor*. If we were to apply the linear drag model to our oscillating cart in an attempt to *describe* the motion, Newton’s Second Law of Motion tells us that *Fnet = -kx - bv = ma*, where *k* is the effective spring constant *kA* + *kB*. This can be rewritten as a second-order differential equation as follows:

The solution to this differential equation, ***where b is small***, is given by

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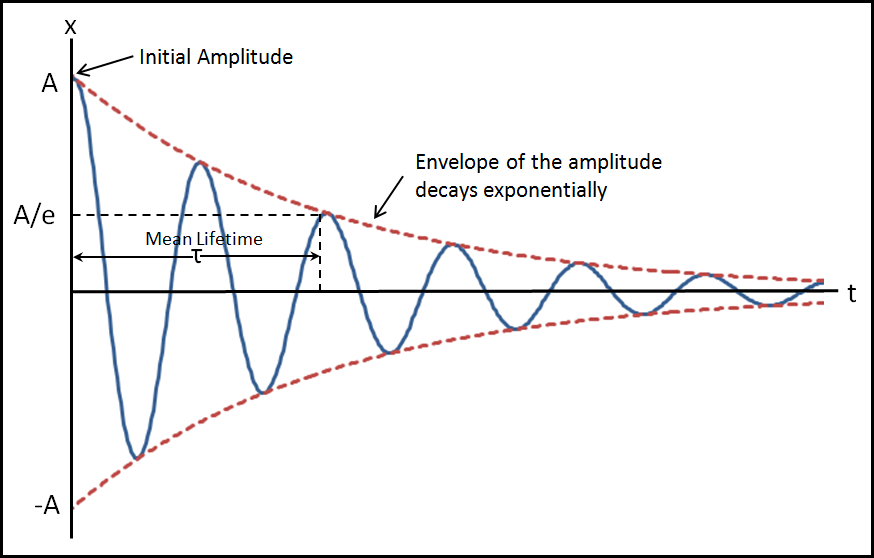
where *A* is the *initial amplitude*, and is the *phase constant*. The phase constant can be increased or decreased by any integral multiple of 2π (or 360°), and the motion will be described equally well. The angular frequency *ω* is given by

where *f* is the frequency and *T* is the period for the oscillations.

If there were no damping (*b* = 0), then the amplitude would remain constant and the angular frequency would be , which is the angular frequency for an undamped oscillator. When damping is present (*b* > 0), then ω is less than that for the undamped oscillator.

See Figure 4 to aid in visualizing the following discussion. The exponential term in the solution tells us that the amplitude of the oscillation gradually decreases, approaching zero as time progresses. This is shown by the envelop of red dashed lines in Figure 4. The amount of time *τ* required for the amplitude to decrease to 1/e (≈0.37 or 37%) of its original value is known as the *mean lifetime* of the oscillation. Since the amplitude factor of the solution is , then *τ* = 2*m*/*b*. Increasing the value of *b* will result in a shorter mean lifetime, while decreasing *b* will provide for a longer mean lifetime. From the defining equation for *b*, it is seen that the units for *b* must be N/(m/s), or equivalently from *τ* = 2*m*/*b*, the units of *b* are simply kg/s. Thus the argument –*bt*/2*m* of the exponential function is a pure number without units, as it should be.

Any phenomenon that decays exponentially over time will decay to 37% of its initial value after one average lifetime *τ* has transpired. This occurs in a large variety of systems in the world of science and engineering and is something that physics students should know and understand well.

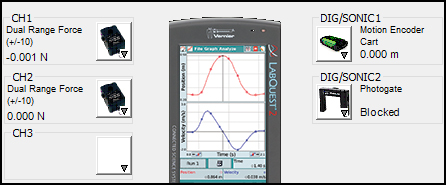


*Figure 4*

***Data Collection***

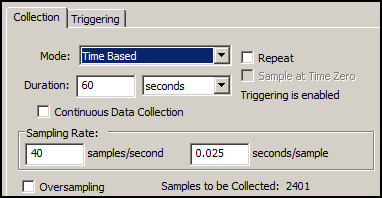
After setting up the apparatus as described earlier and with LabQuest 2 connected to your computer, you can begin thinking about how to collect the data. With the file *Oscillating Cart Template.cmbl* opened on your computer, the first thing you should do is make sure to record the mass of the cart and the spring constants in the parameter controls on the left side of the screen. If they are significantly different than the default values shown, you can select *Data/User Parameters* from the Logger *Pro* main menu and enter the new values directly rather than use the up and down arrow buttons on the parameter controls to change the parameter values.

Next you should check that all of the sensors are set-up correctly. Select *Experiment/Set Up Sensors/Show All Sensors* from the Logger *Pro* main menu. Figure 5 shows a portion of the resulting screen. The cart should now be resting at its equilibrium position. Click the down-arrow for analog CH1 (the left Dual Range Force Sensor), and make sure to *Zero* it and that it is set for *Reverse*. Analog CH2 should be connected to the Dual-Range Force Sensor on the right—you should *Zero* it, but it should NOT be set to *Reverse*. DIG1 should be connected to the Motion Encoder, set to *Reverse*, and you should select *Zero* when the cart is at its equilibrium position. (Make sure that you have turned the Motion Encoder Cart on by pressing the clear power button on the cart endcap!) Finally, click the down arrow for DIG2, which should be connected to the Photogate. It should be set to *Pendulum* timing. With the cart at rest at the equilibrium position it should show as being blocked.



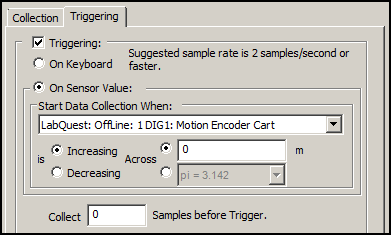
*Figure 5*

Just one more thing to double-check and you can begin data collection! Select *Experiment/Data Collection* from the Logger *Pro* main menu. The *Collection* tab (see Figure 6) should indicate that the *Mode* is *Time Based*, with a *Sampling Rate* of 40 samples/second. This rate works well and can be left as is. The *Duration* defaults at 60 seconds, which is fine when both springs have 5 N/m spring constants. You will probably want to reduce the duration for other combinations of springs in which damping is stronger in order to avoid data collection when the amplitude is negligible or small at the end of the run.



*Figure 6*

The Triggering tab (see Figure 7) should show that *Triggering* is turned on and that data collection will start when the Motion Encoder Cart is *Increasing Across 0 m*. This means that you should pull the cart toward the left end of the track, click the *Start Collection* button on the Logger *Pro* toolbar, and then release the cart and let it oscillate. When you are satisfied with the run, you can save your file, but don’t save it by overwriting the *Oscillating Cart Template.cmbl* file. Always begin with the clean template file when changing springs or changing to a cart with a different mass.



*Figure 7*

With all of your data collected you are ready for the most exciting part of this experiment—data analysis!

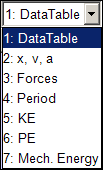
***Data Analysis***

Our discussion of the data analysis is based upon a heavy cart set up as shown in Figure 8. The 500 g mass is attached to the cart with the tall aluminum post above it to allow for pendulum timing with the Photogate. Two short aluminum posts are mounted on either side of the cart for attaching the springs. The mass of the cart as set up was 1.115 kg.



*Figure 8*

Spring A (the left spring) was a 5 N/m spring, while Spring B (the right spring) was a 15 N/m spring. The file ***Heavy Cart A5N B15N Springs.cmbl*** is part of the documentation supporting this investigation and contains the data immediately following data collection prior to any analysis. This file will be used in our discussion of data analysis. You will notice after opening this file that there are seven pages available for analysis, with the file opening by default on Page 1, containing the data table. If you click on the drop-down arrow for selecting pages in the toolbar, you will see the page choices as shown in Figure 9.

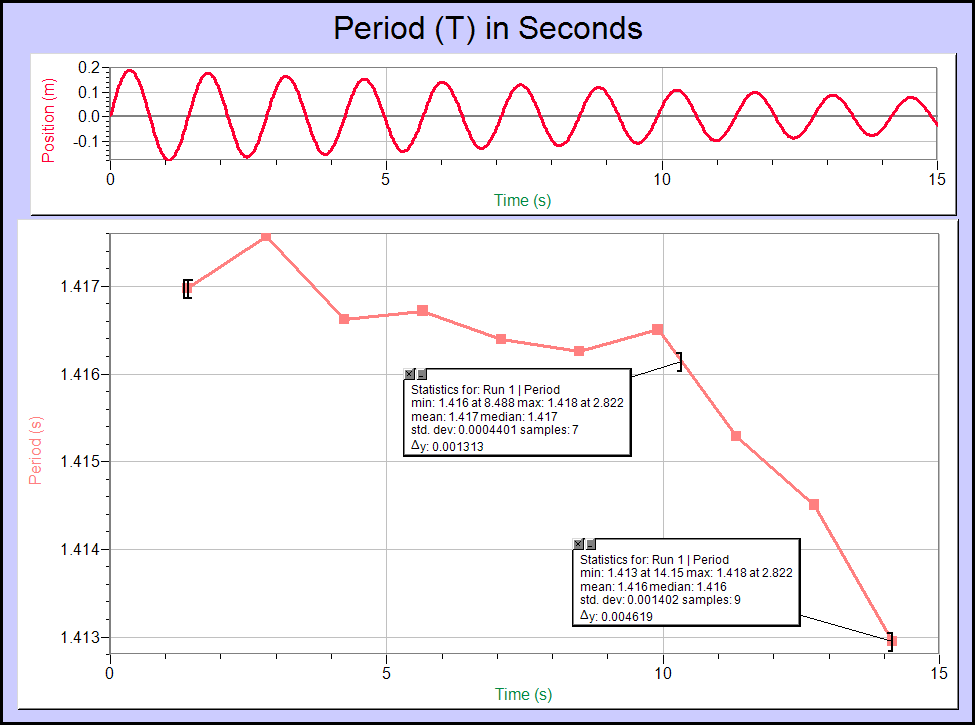


*Figure 9*

We begin the analysis by studying *Page 4: Period*, as the period of the oscillating cart will be needed at a number of places in the analysis.

*Page 4: Period*

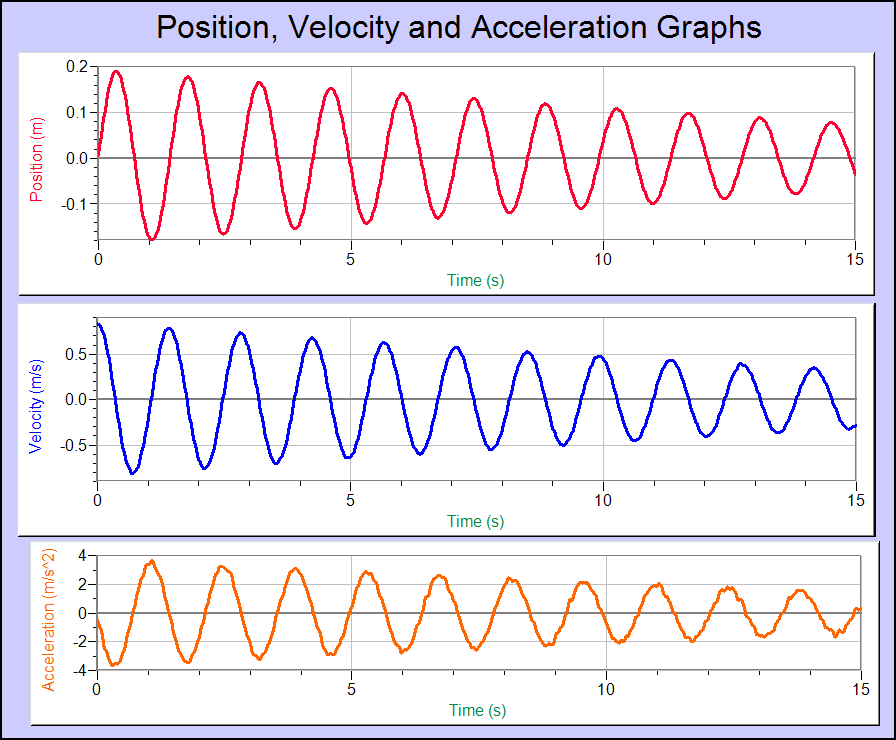
Figure 10 shows what the Period *vs*. Time graph looks like and allows for easy comparison with the Position *vs*. Time graph. Points for period have been plotted automatically, with a point appearing every other time that the cart crosses the photogate (located at position zero), in accordance with pendulum timing. It is seen after applying *Analyze/Statistics* to the entire data range that the mean period for the oscillating cart is 1.416 seconds. It is noted that the period seems to drop off more rapidly as the amplitude (position of the cart) decreases over time. If you wanted to, you could analyze statistics for all but the last three points. In this case you get a slightly higher mean period of 1.417 seconds.



*Figure 10*

*Page 2: x, v, a (Position, Velocity Acceleration)*

This page provides graphs of position, velocity, and acceleration versus time, allowing easy comparison of their relationships, as shown in Figure 11. Since velocity and acceleration graphs are produced automatically when using the Motion Encoder System, it is a good idea to shown students the formulas that are used for computing velocity and acceleration. These can be quickly seen by clicking *Data/Column Options* from the main Logger *Pro* menu and then selecting *Velocity* or *Acceleration*. The expression for Velocity is **derivative("Position", "Time")** and it computes the weighted average of the slope of *n* points around each point. The expression for Acceleration is **secondDerivative("Position", "Time")**, and it computes the second derivative of Position with respect to Time. The value of *n* can be set by selecting *File/Settings for (file name)* from the main menu in Logger *Pro*. The value of *n* for this experiment is the default value of 7.

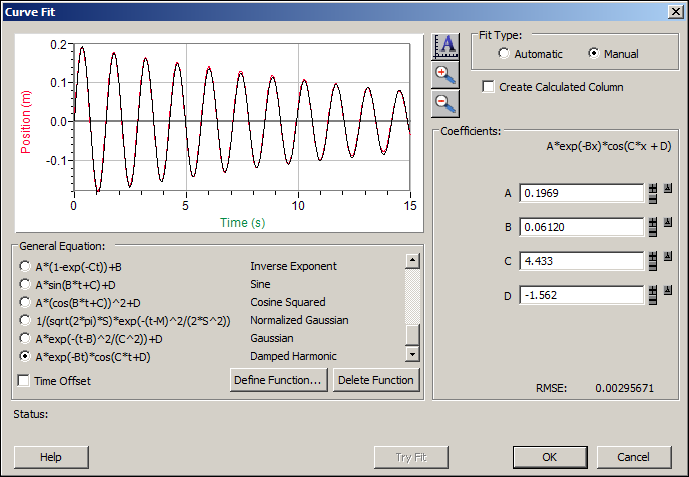


*Figure 11*

In addition to the sinusoidal variation in the position of the oscillating cart, it is quickly observed that the envelop of the amplitude seems to be decreasing, perhaps in a negative exponential fashion. This suggests trying the following curve in an effort to model the position data for the oscillating cart:

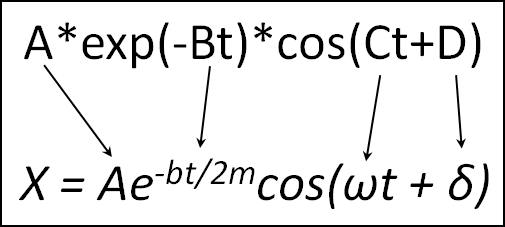
A\*exp(-Bt)\*cos(C\*t+D)

With the Position *vs*. Time graph selected, click on *Analyze/Curve Fit* from the main Logger *Pro* menu. Since this function is not one of the built-in functions, click on the *Define Function* button in the *Curve Fit* dialog. Key in **A\*exp(-Bt)\*cos(C\*t+D)**for *f(t)* and **Damped Harmonic** for the *Description*. Then click the *Try Fit* button in the *Curve Fit* dialog. After a few seconds a *Curve Fit* dialog will appear similar to what is shown in Figure 12. The fit of our position data to the damped harmonic model is very good! The magnitude of the A coefficient should be close to the maximum amplitude x. The magnitude of the coefficients B and C will agree with those shown in Figure 12, though the signs may differ—no problem, as the magnitudes of these coefficients are of most interest. The coefficient D may differ from what is shown in Figure 12. However, as mentioned in the section *A Model for Damping*, you can add or subtract any integral multiple of 2π and still have a perfect fit. You can try this if you wish by clicking the *Set Delta* button for the D coefficient to, say, π/2=1.5708. After clicking the ***+*** button or the ***–*** button four times (equal to 2π) in a row, the fitted curve will again line up with the experimental position data for the oscillating cart!



*Figure 12*

Now we get to the most interesting of the coefficients as they relate to our damped harmonic oscillator—the B and C coefficients. Figure 13 shows the relationship of these coefficients to the model equation for damping. The coefficient B corresponds to b/2m, allowing us to calculate the value of the *damping factor b*. The coefficient C corresponds to the angular frequency *ω = 2πf = 2π/T*, giving us an indirect way to determine the period T. After a little arithmetic, we find that b = 2mB = 2(1.115kg)(0.0612/s) = 0.136 kg/s and T = 2π/C = 2π/(4.433/s) = 1.417 s. We have thus obtained the value of the ***damping factor b*** as well as a value for the ***period T*** that agrees with the direct observation obtained from the Photogate. Finally, we can calculate the mean lifetime *τ* = 2*m*/*b = 1/B = 1/(0.0612/s) = 16.3 s,* the time required for the amplitude *x* of the oscillating cart to decrease to 37% of its initial value*.*



*Figure 13*

*Page 3: Forces (Spring Forces)*

Figure 14 shows the graph of the spring forces acting on the cart as measured by the two Dual-Range Force Sensors. It is seen that when the cart is on the right (+x position), both Spring A and Spring B have negative forces, in effect pulling the cart to the left. (Don’t forget that we zeroed the force sensors when at equilibrium.) When the cart is on the left (-x position), both springs have positive forces, in effect pulling the cart to the right. The sum of the spring forces is also plotted on the graph in green.

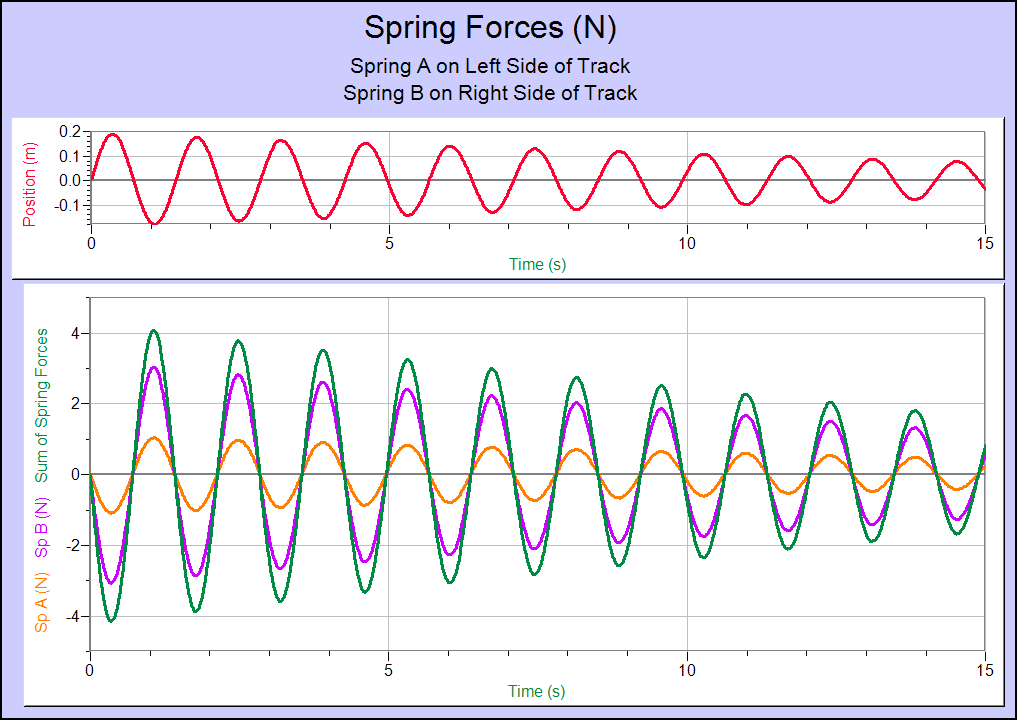
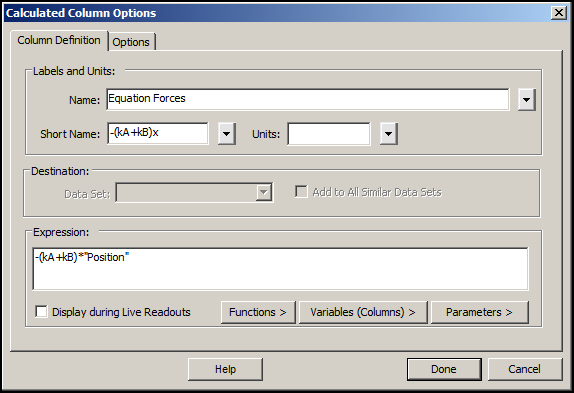


Figure 14

We learned earlier in this document that the two separate springs with spring constants *kA* and *kB* act like a single spring whose spring constant is *kA + kB*, giving a net spring force of *–(kA +kB)x*. We would expect a plot of *–(kA +kB)x* to coincide fairly well with the Sum of Forces green line on the chart of Figure 14. To do this, click on *Data/New Calculated Column* in the Logger *Pro* main menu and fill in information as shown in Figure 15.

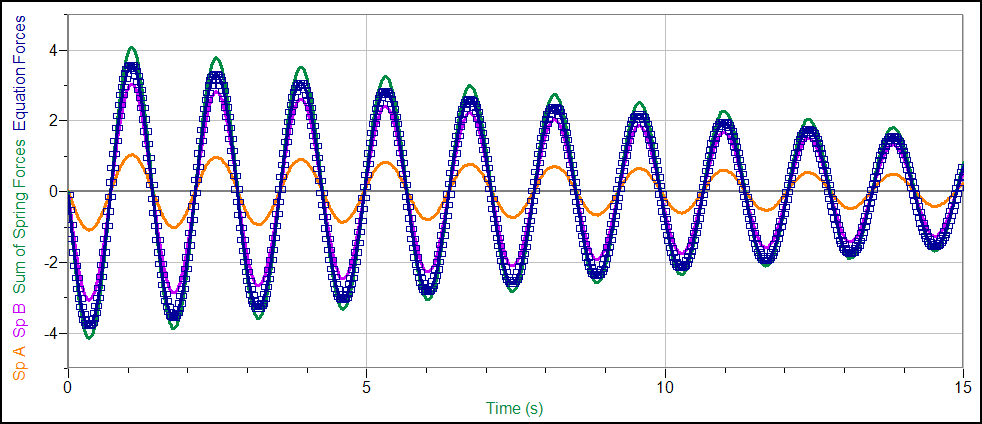


*Figure 15*

To get this new calculated column to appear on the graph:

* Right click on the forces graph and select *Graph Options.*
* Click on the *Axes Options* tab in the *Graph Options* dialog.
* Click on the *Equation Forces* item in the *Y-Axis Columns* text area so that it is checked.
* Click the *Done* button on the *Graph Options* dialog.

Your forces graph should now appear about as shown in Figure 16, where the blue line shows the net spring forces as predicted from the equation *–(kA +kB)x*. You may be able to improve the fit somewhat if you use measured spring constants for the two springs rather than accept the nominal values in their packaging. The author found that the nominal 15 N/m spring was actually more like 16.2 N/m.



*Figure 16*

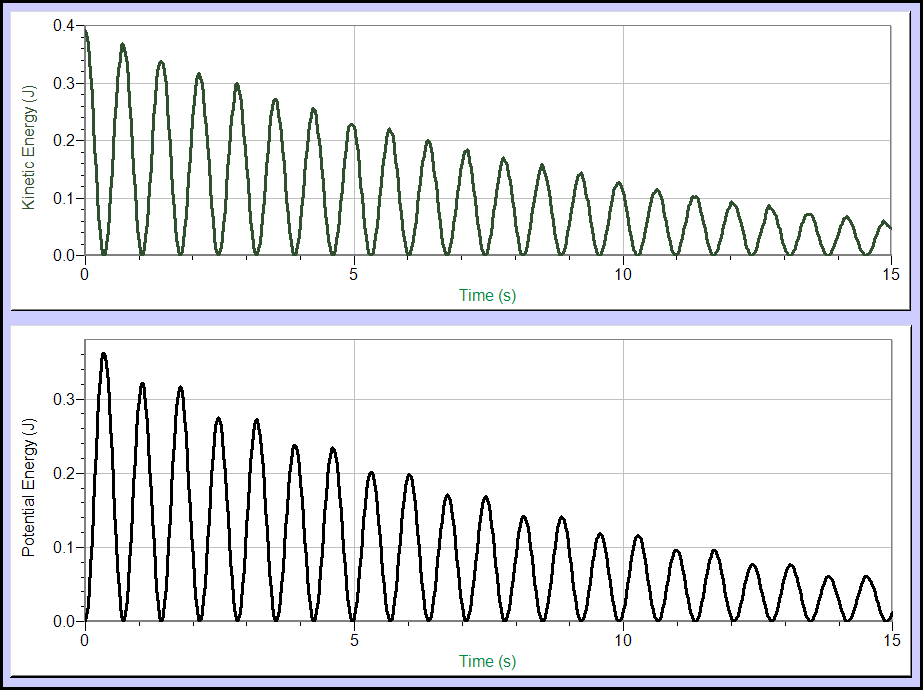
*Pages 5-6: KE, PE, (Kinetic Energy, Potential Energy)*

Based upon the defining equation for kinetic energy, *KE = ½mv2*, and the equation for potential energy for a spring, *PE = ½kx2*, you can view the expressions used to obtain our graphs for *KE* and *PE*:

0.5\*m\*"Velocity"^2

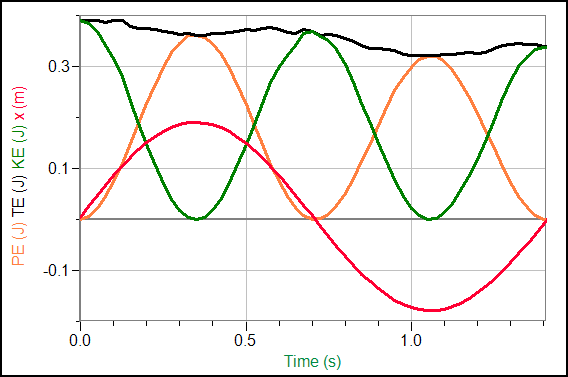
0.5\*kA\*"Position"^2+0.5\*kB\*"Position"^2

The graphs obtained for *KE* and *PE* are similar, as shown in Figure 17. Both start off at time zero at approximately 0.4 J, both have envelopes that show energy dropping off in what appears to be a negative exponential fashion, and both show a uniform periodic variation. However, they are 180° out-of-phase with one-another, as one would expect as energy is cyclically converted between potential energy stored in the springs and kinetic energy of the moving cart.



*Figure 17*

It is instructive to have students prepare a graph showing *PE*, *KE*, Mechanical Energy, and Position all on the same graph, but just for the first full cycle of the back-and-forth motion of the cart. The graph will appear similar to that shown in Figure 18.



*Figure 18*

The 180° phase difference between the *KE* and *PE* graphs is now much clearer. In addition, it is seen the period of oscillation for both *KE* and *PE* is half that of the period of the cart’s oscillation back-and-forth on the track—expected since each reaches a maximum twice for each complete cart oscillation. The black line on the chart shows the mechanical energy, which is the sum of the KE and PE.

*Page 7: Mech. Energy (Mechanical Energy)*

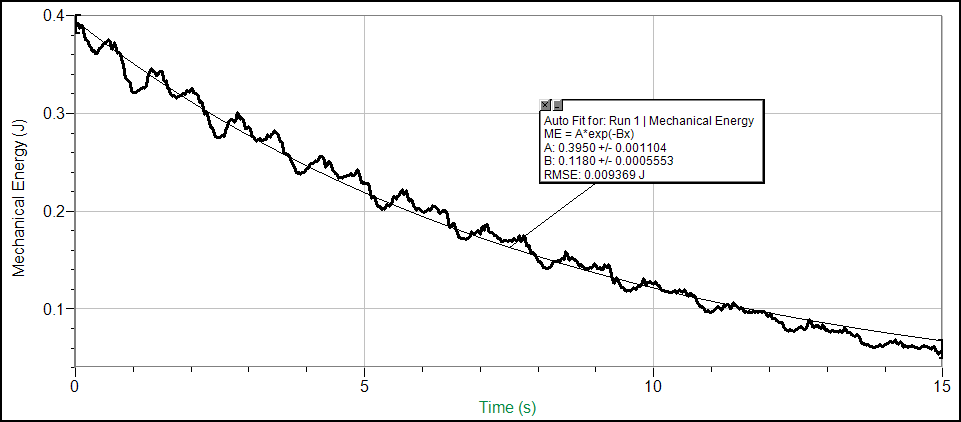
Mechanical energy for this system is defined by the sum of *KE* and *PE*. Therefore, the expression for Mechanical Energy in Logger *Pro* is

"Potential Energy"+"Kinetic Energy"

Figure 19 shows the graph of mechanical energy *vs*. time. It appears to be decreasing exponentially with time. The students should be asked to verify this observation by using *Analyze/Curve Fit* to try the function

A\*exp(-Bt)

This is not a built-in function in Logger *Pro*, so students will need to define a custom function in a manner similar to the way this was done when modeling position earlier in our analysis. Figure 19 also shows the result of applying this function to the mechanical energy data.



*Figure 19*

The coefficient A is 0.395 J, which is the initial ME at time zero, clearly seen on the graph. The mean lifetime τ = 1/B = 1/(0.118/s) = 8.47 seconds. This is very close to ***half*** the mean lifetime (16.3 seconds) of the amplitude of the oscillating cart’s position calculated earlier. This is not a coincidence! It is a result of the fact that *PE* is proportional to *x*2 and *KE* is proportional to *v*2. A good exercise for the student is to take the equation for position *x(t)*

A\*exp(-Bt)\*cos(C\*t+D)

and square it. The resulting exponential becomes *exp(-2Bt)*, which exponentially drops off twice as fast as *exp(-Bt).* Students who know differential calculus can compute the derivative of *x(t)* with respect to *t* to get the velocity. Squaring the resulting equation for velocity will again result in the exponential *exp(-2Bt)*. *For damped harmonic motion, the mean lifetime for mechanical energy is half the mean lifetime for the amplitude x of the oscillation*.

***Cart and Spring Combinations***

You may want to have each lab group do a variety of runs with different cart/spring combinations, or have each lab group select one cart/spring combination for their experiment. The table in Figure 20 lists a few of the possibilities.

|  |  |  |
| --- | --- | --- |
| **Cart** | **A (Left Side of Cart)** | **B (Right Side of Cart)** |
| Regular | 5 N/m Spring | 5 N/m Spring |
| Regular | 15 N/m Spring | 15 N/m Spring |
| Regular | 15 N/m Spring | 5 N/m Spring |
| Regular | ~25 N/m Stretchable 7” Bands (from office supply store) | ~25 N/m Stretchable 7” Bands (from office supply store) |
| Heavy (added 500 gm mass) | 5 N/m Spring | 5 N/m Spring |
| Heavy (added 500 gm mass) | 15 N/m Spring | 15 N/m Spring |
| Heavy (added 500 gm mass) | 15 N/m Spring | 5 N/m Spring |
| Heavy (added 500 gm mass) | ~25 N/m Stretchable Bands (from office supply store) | ~25 N/m Stretchable Bands (from office supply store) |

*Figure 20*

If your students are familiar with the law for determining spring constants when they are in series, *k-effective = kAkB/(kA + KB)*, you can add many more combinations of cart/springs for the experiment. Even if they don’t know this law, they can measure *k-effective* experimentally by stretching the series combination a measured distance while it is connected to the cart and Dual-Range Force Sensor. Actually, it is probably a good idea to always measure the spring constants this way prior to doing this damped harmonic motion experiment, as spring constants may change with use over time. One of the best ways to connect two springs together in series is to use a small paper-clip. Make sure to remind students not to stretch the springs beyond the limits suggested in the *Oscillating Cart System Forces* section of this document. Also, as suggested earlier, students should wear goggles since stretched springs may snap.

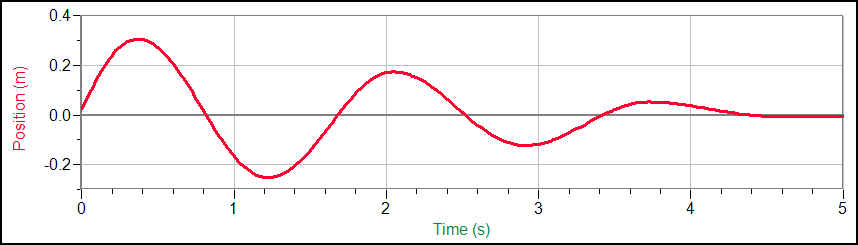
***Additional Investigation***

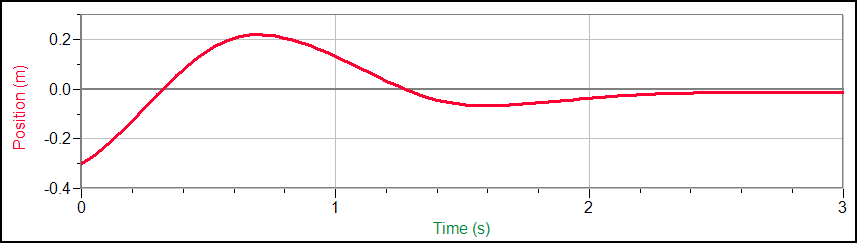
We saw earlier that for the damped harmonic oscillator

The quantity *Q* inside the square root radical suggests three possible scenarios:

1. *Q* > 0 implies *b2 < 4mk* and we have ***underdamping*** in which *b* is small when compared to *m* and *k*, studied so far in this investigation.
2. *Q* < 0 implies *b2 > 4mk* and we have ***overdamping***, in which *b* is large when compared to *m* and *k*.
3. *Q* = 0 implies *b2 = 4mk* and we have ***critical damping***, separating underdamping and overdamping.

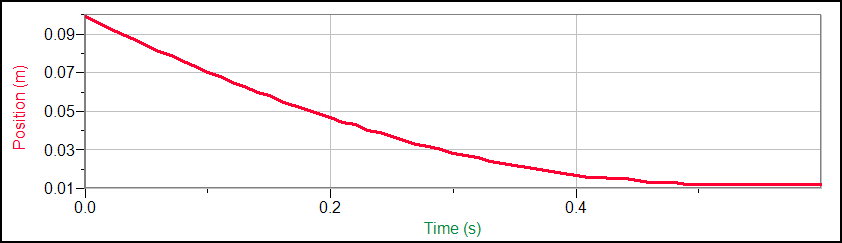
Challenge your students to do experiments using the *Cart Friction Pad* in a way that will increase *b* enough to produce results such as those shown in Figure 21.





*Figure 21*

Challenge your students further to produce overdamping. Here the motion will not be periodic—it will simply return to its equilibrium position when released. The position versus time graph will appear similarly to what is shown in Figure 22.



*Figure 22*

[The author accomplished this by using a heavy cart with the Cart Friction Pad and a 5 N/m spring on either side of the cart. The cart was released only a short distance from its equilibrium position.]

***Question for Class Discussion***

Storm doors often have a damped spring, either at the top or bottom of the door, to control how fast the door closes. What type of damping is generally desired for such a door—underdamped or overdamped? What about a typical door that separates the kitchen from the serving area in a restaurant?